Well-posedness and long time behavior of singular Langevin SDEs

Renming Song

University of Illinois

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References

This talk is based on the following joint paper with Longjie Xie:

Well-posedness and long time behavior of singular Langevin stochastic differential equations. arXiv: 1804.05086

Outline

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The stochastic Langevin equation

$$\begin{cases} dX_t = V_t dt, & X_0 = x \in \mathbb{R}^d \\ dV_t = -\gamma V_t dt - \nabla F(X_t)dt + dW_t, & V_0 = v \in \mathbb{R}^d \end{cases}$$

describes the motion of a particle in a smooth potential field, subject to friction and stochastic forcing.

Here W is a d-dimensional standard Brownian motion, $\gamma > 0$ is a friction constant (which ensures dissipation), and $F : \mathbb{R}^d \to \mathbb{R}$ is a non-negative smooth function.

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The Langevin equation above is degenerate since the noise and the dissipation only appear in the momentum part. However, due to the interaction between position and momentum, noise and dissipation get transmitted from the momentum part to the position part, which ultimately leads to ergodicity and exponential convergence to equilibrium.

For some literature on stochastic Langevin equation, see Bou-Rabee and Sanz-Serna (*AAP*, 2017); Cerrai and Freidlin (*J. Stat.Phys*, 2015); Eberle and Guillin (2017); Hairer and Mattingly (*CMP* 2009); Ottobre and Pavliotis (*JFA*, 2012); Wu (*SPA*, 2001); . . . .

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The generator of the stochastic Langevin equation is $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$, where

$$\mathcal{L}_1 := \frac{1}{2}\Delta_v - \gamma v \cdot \nabla_v$$

is the generator of an OU process and

$$\mathcal{L}_2 := v \cdot \nabla_x - \nabla F(x) \cdot \nabla_v$$

is the Liouville operator associated with the Hamiltonian $H(x, v) := \frac{1}{2}|v|^2 + F(x)$.

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Under the condition that there exist constants $C_0 > 0$ and $\vartheta \in (0,1)$ such that

$$\frac{1}{2}\langle x, \nabla F(x)\rangle \geq \vartheta F(x) + \gamma^2 \frac{\vartheta(2-\vartheta)}{8(1-\vartheta)}|x|^2 - C_0, \tag{1}$$

one can check that

$$\mathcal{H}(x,v) := H(x,v) + \tfrac{\gamma}{2}\langle x, v\rangle + \tfrac{\gamma^2}{4}|x|^2 + 1 \tag{2}$$

is a good choice of a Lyapunov function, i.e., for some positive constants c_0, K_1 and all $x, v \in \mathbb{R}^d$,

$$\mathcal{H}(x,v) \geq 1 + \tfrac{\gamma^2}{12}|x|^2 + \tfrac{1}{8}|v|^2 \quad \text{and} \quad \mathcal{L}\mathcal{H}(x,v) \leq -c_0\mathcal{H}(x,v) + K_1.$$

Recently, stochastic Langevin equations with singular potential filed F have been studied, see Conrad and Grothaus (*J. Evol. Eq.*, 2010); Cooke, Herzog and Mattingly (*Comm. Math. Sci.* 2017) and Grothhaus and Stilgenbauer (*Integr. Eq. Op Th* 2015).

The exponential ergodicity was also obtained by constructing explicit Lyapunov functions.

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The exponential ergodicity was also obtained by constructing explicit Lyapunov functions.

We study the existence and uniqueness of strong solutions of the stochastic Langevin equation

$$\begin{cases} dX_t = V_t dt, & X_0 = x \in \mathbb{R}^d \\ dV_t = -\gamma V_t dt - \nabla F(X_t)dt + G(V_t)dt + dW_t, & V_0 = v \in \mathbb{R}^d \end{cases}$$

under the presence of a singular velocity field G. We also study the exponential ergodicity of this stochastic Langevin equation. The singular velocity field destroys the dissipation in the momentum part and makes the classical Lyapunov condition very difficult to check, if possible at all.

Outline

Before stating our main result, we first recall the following definition

Definition

Suppose $\mathcal{H}$ is a positive function on $\mathbb{R}^{2d}$. The invariant distribution μ (if exist) of an $\mathbb{R}^{2d}$-valued Markov process M_t is said to be $\mathcal{H}$-uniformly exponentially ergodic, if there exist constants $C, \eta > 0$ such that for all $y \in \mathbb{R}^{2d}$ and all Borel functions $f : \mathbb{R}^{2d} \to \mathbb{R}^d$ with $|f| \leq \mathcal{H}$,

$$\left|\mathbb{E}^y f(M_t) - \mu(f)\right| \leq C\mathcal{H}(y)e^{-\eta t}, \quad \forall t \geq 0,$$

where $\mathbb{E}^y$ is the expectation with respect to $\mathbb{P}^y$, the law of M with initial value $M_0 = y$, and $\mu(f)$ denotes the integral of f respect to μ.

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Our main result is

Theorem 1

Let $G \in L^p(\mathbb{R}^d)$ with $(2 \vee d) < p \leq \infty$. For each $y := (x, v)^{\mathrm{T}} \in \mathbb{R}^{2d}$, our stochastic Langevin equation admits a unique strong solution $Y_t = (X_t, V_t)^{\mathrm{T}}$. Moreover, if we further assume that (1) holds together with one of the following conditions:

(A) $|\nabla F(x)|^2 \leq C_1(1 + |x|^2 + F(x))$;

(B) $|\nabla F(x)| \leq C_1(1 + |x|^2 + F(x))$ and $G \in L^p(\mathbb{R}^d)$ with $2d < p \leq \infty$,

where $C_1 > 0$ is a constant, then Y_t has a unique invariant distribution μ which is $\mathcal{H}$-uniformly exponentially ergodic with $\mathcal{H}$ being the function defined in (2).

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Condition (**A**) includes the case of a harmonic potential $F(x) = \frac{1}{2}|x|^2$, while any polynomial F which grows at infinity like $|x|^{2\ell}$ for some positive integer ℓ satisfies condition (**B**). The presence of the singular term G can be used to describe stochastic non-linear oscillators as well as degenerate particle systems arising in math physics. Here is a particular example.

$$\begin{cases} dX_t = V_t dt, & X_0 = x \in \mathbb{R}^d \\ dV_t = -\gamma V_t dt - X_t dt + \frac{1}{|V_t|^\alpha} 1_{|V_t| \le K} dt + dW_t, & V_0 = v \in \mathbb{R}^d \end{cases}$$

where $K > 0$ and $\alpha \in (0, 1)$ are constants.

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where $K > 0$ and $\alpha \in (0, 1)$ are constants.

The main difficulty for studying our stochastic Langevin equation with singular velocity field that we have to treat simultaneously the singular term G and the super-linear growth part F in the coefficients. Here is a brief description of our strategy.

To prove the well-posedness of the stochastic Langevin equaton, we will use Zvonkin's transform combined with a local Krylov's estimate and a localization technique. The super-linear growth of F makes deriving our local Krylov's estimate pretty challenging.

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To study the long time behavior, the localization technique is of no help. This is why we need some growth conditions on F to derive a global Krylov's estimate for the solution. Conditions (**A**) and (**B**) reflect that some balances are needed between the integrability of G and the growth property of F.

In a recent paper by Xie and Zhang, ergodicity of non-degenerate SDE's with singular dissipative drifts was established by using Zvonkin's transform to remove the singular drift and the fact that dissipativity is preserved by Zvonkin's transform.

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Such an idea does not work for stochastic Langevin equation. The Hamiltonian structure will completely destroyed by Zvonkin's transform. It is very difficult, if possible at all, to find a Lyapunov function for the equation.

To overcome this difficulty, we will use Krylov's estimate to get a good control on the expectation of the singular part, and then combine with the Lyapunov technique to get the existence of invariant distributions. The uniqueness of invariant distribution follows by the strong Feller property and irreducibility of the unique strong solution.

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Outline

For every $f \in L^p(\mathbb{R}^d)$ with $p > (d \vee 2)$, we have

$$\sup_{v \in \mathbb{R}^d} \mathbb{E} \exp \left\{ \int_0^t |f(v + W_s)|^2 ds \right\} \leq C_d \|f\|_p e^{C_d t},$$

where $C_d > 0$ is a constant. Using Girsanov's theorem, we can prove the following local Krylov's estimate.

Lemma 2

Assume that $G \in L^p(\mathbb{R}^d)$ with $p > (d \vee 2)$. Then, for every initial value $y = (x, v)^{\mathrm{T}} \in \mathbb{R}^{2d}$, there exists a weak solution to the stochastic Langevin equation. Moreover, let $(X_t(x), V_t(v))^{\mathrm{T}}$ solve the stochastic Langevin equation and for every $R > 0$, define

$$\tau_R^v := \inf\{t \geq 0 : |V_t(v)| \geq R\}. \tag{3}$$

Then, for every $T > 0$ and $f \in L^q(\mathbb{R}^d)$ with $q > (d/2) \vee 1$, we have

$$\mathbb{E} \exp\left\{\int_0^{T \wedge \tau_R^v} f(V_s(v))ds\right\} \leq C_R e^{C_R T},$$

where $C_R = C(d, x, v, R, \|f\|_q, \|G\|_p)$ is a positive constant which is uniformly bounded for (x, v) in compact sets and $C_R \to 0$ as $\|f\|_q \to 0$.

We also show that that for every $f \in L^q(\mathbb{R}^d)$ with $q > (d/2) \vee 1$, it holds that

$$\mathbb{E}\left(\int_0^{T\wedge\tau_R^v} f(V_s(v))ds\right) \leq C_R e^{C_R T},$$

where $C_R \to 0$ as $\|f\|_q \to 0$.

To derive our global Krylov's estimate, we will need to consider the following quasi-linear elliptic equation:

$$\lambda u(x) - \frac{1}{2}\Delta u(x) - G(x) \cdot \nabla u(x) - \kappa|\nabla u(x)|^2 = f(x), \qquad (4)$$

where $\lambda, \kappa \geq 0$ are constants.

The following result will be used to prove our global Krylov's estimate. The key point of the proof of the following result is to use Sobolev's embeddings to handle the non-linear term $|\nabla u|^2$.

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Lemma 3

Let $\kappa \geq 0$ be a constant.

(i) Suppose that $G \in L^p(\mathbb{R}^d)$ with $p > d$. Then for every $f \in L^q(\mathbb{R}^d)$ with $q > d/2$, there exists a $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, (4) admits a unique solution $u \in W^{2,q}(\mathbb{R}^d)$. Moreover, we have

$$\lambda \|u\|_q + \sqrt{\lambda}\|\nabla u\|_q + \|\nabla^2 u\|_q \leq C_1 \|f\|_q,$$

where $C_1 = C(d, \|G\|_p) > 0$ is a constant.

(ii) Let p and q be as in part (i). Given two sequences of functions $G_n, f_n \in C_0^\infty(\mathbb{R}^d)$ such that $G_n \to G$ in $L^p(\mathbb{R}^d)$ and $f_n \to f$ in $L^q(\mathbb{R}^d)$. Let u_n be the corresponding solution to (4) with G, f replaced by G_n, f_n. Then we have $u_n \in C_b^2(\mathbb{R}^d) \cap W^{2,q}(\mathbb{R}^d)$ with

$$\sup_{n \geq 1} \|u_n\|_{2,q} \leq C_2 \quad \text{and} \quad \|u_n - u\|_{2,q} \leq C_2\big(\|f_n - f\|_q + \|G_n - G\|_p\big),$$

where $C_2 = C(d, \|G\|_p, \|f\|_q)$ is a positive constant.

Here is our global Krylov's estimate under (**A**).

Lemma 4

Assume that $G \in L^p(\mathbb{R}^d)$ with $p > (d \vee 2)$ and that condition **(A)** holds. Let $(X_t(x), V_t(v))^{\mathrm{T}}$ solve our stochastic Langevin equation. Then, for any $f \in L^q(\mathbb{R}^d)$ with $q > (d/2) \vee 1$, there exists a constant $C = C(d, x, v, \|G\|_p, \|f\|_q) > 0$ such that for all $t \geq 0$,

$$\mathbb{E}\left(\int_0^t f(V_s)ds\right) \leq C(1+t).$$

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Lemma 4

Assume that $G \in L^p(\mathbb{R}^d)$ with $p > (d \vee 2)$ and that condition **(A)** holds. Let $(X_t(x), V_t(v))^{\mathrm{T}}$ solve our stochastic Langevin equation. Then, for any $f \in L^q(\mathbb{R}^d)$ with $q > (d/2) \vee 1$, there exists a constant $C = C(d, x, v, \|G\|_p, \|f\|_q) > 0$ such that for all $t \geq 0$,

$$\mathbb{E}\left(\int_0^t f(V_s)ds\right) \leq C(1+t).$$

Here is our global Krylov's estimate under (**B**).

Lemma 5

Assume that condition **(B)** holds and that $(X_t(x), V_t(v))^{\mathrm{T}}$ solves our stochastic Langevin equation.. Then, for every $f \in L^q(\mathbb{R}^d)$ with $q > d$, there exists a constant $C = C(d, x, v, \|G\|_p, \|f\|_q) > 0$ such that

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Outline

To use Zvonkin's transform to remove the singular term $G(V_t)$, we let u be the solution to the following equation

$$\lambda u(x) - \frac{1}{2}\Delta u(x) - G(x)\cdot\nabla u(x) = G(x). \tag{5}$$

Since $G \in L^p(\mathbb{R}^d)$ with $p > d$, we have $u \in W^{2,p}(\mathbb{R}^d)$. By the Sobolev embedding, we can get that $u \in C_b^1(\mathbb{R}^d)$.

Lemma 6

Let $Y_t = (X_t, V_t)^{\mathrm{T}}$ solve our stochastic Langevin equation and τ_R^v be given by (3). Then we also have that for $t \le \tau_R^v$,

$$\begin{cases} dX_t = V_t dt, \\ dV_t = [-\gamma V_t - \nabla F(X_t)]\sigma(V_t)dt + [\lambda u(V_t)dt - du(V_t)] + \sigma(V_t)dW_t, \end{cases} \tag{6}$$

with initial value $(x, v)^{\mathrm{T}}$, where

$$\sigma(v) := \big(\mathbb{I} + \nabla u(v)\big).$$

In the new SDE (6), the singular drift disappears, but the Hamiltonian structure has been totally destroyed. Thus it is difficult to find a Lyapunov function for (6).

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Proof of strong well-posedness

Weak existence has been given in Lemma 2. By the Yamada-Watanabe principle, it suffices to show the pathwise uniqueness. Consider two solutions $Y_t(y) := \big(X_t(x), V_t(v)\big)^{\mathrm{T}}$ and $Y_t(\hat{y}) := \big(X_t(\hat{x}), V_t(\hat{v})\big)^{\mathrm{T}}$, defined on the same probability space and with respect to the same Brownian motion, starting at $y := (x, v)^{\mathrm{T}}$ and $\hat{y} := (\hat{x}, \hat{v})^{\mathrm{T}}$, respectively. Define

$$\zeta_R := \inf\{t \geq 0 : |V_t(v)| \vee |V_t(\hat{v})| \geq R\}.$$

Let us fix $T > 0$ below. We proceed to prove that for every $\beta \in (0, 1)$, $t \leq T$ and $R > 0$, there exists a constant $C_{R,T} > 0$ such that for all $y, \hat{y} \in \mathbb{R}^{2d}$ with $|y|, |\hat{y}| \leq R$,

$$\mathbb{E}|Y_{t\wedge\zeta_R}(y) - Y_{t\wedge\zeta_R}(\hat{y})|^{2\beta} \leq C_{R,T}|y - \hat{y}|^{2\beta}. \tag{7}$$

Once this is proved, we can apply the special case $y = \hat{y}$ to get the pathwise uniqueness of solutions

Proof of strong well-posedness (cont)

Set $\tilde{X}_t := X_t(x) - X_t(\hat{x})$ and $\tilde{V}_t := V_t(v) - V_t(\hat{v})$. Then, by Lemma 6 we know that the difference process $(\tilde{X}_t, \tilde{V}_t)^{\mathrm{T}}$ satisfies the following equation: for $t \leq \zeta_R$,

$$\begin{cases} d\tilde{X}_t = \tilde{V}_t dt, \\ d\tilde{V}_t = \Big(\lambda\big[u\big(V_t(v)\big) - u\big(V_t(\hat{v})\big)\big]dt - d[u(V_t(\hat{v})) - u(V_t(\hat{v}))]\Big) \end{cases}$$

$$+\xi(\tilde{X}_t, \tilde{V}_t)dt + \big[\sigma\big(V_t(\hat{v})\big) - \sigma\big(V_t(\hat{v})\big)\big]dW_t,$$

with initial value $\tilde{X}_0 = x - \hat{x}$ and $\tilde{V}_0 = v - \hat{v}$, where $\xi(\tilde{X}_t, \tilde{V}_t)$ is defined as

$$[-\gamma V_t(v) - \nabla F(X_t(x))]\sigma\big(V_t(v)\big) - [-\gamma V_t(\hat{v}) - \nabla F(X_t(\hat{x}))]\sigma\big(V_t(\hat{v})\big).$$

Proof of strong well-posedness (cont)

Note that for $T > 0$ and $t \leq T \wedge \zeta_R$,

$$|X_t(x)| \vee |X_t(\hat{x})| \leq R + RT.$$

Since $u \in W^{2,p}(\mathbb{R}^d)$, there exists a constant $C_{R,T} > 0$ such that

$$\xi(\tilde{X}_{t\wedge\zeta_R}, \tilde{V}_{t\wedge\zeta_R}) \leq C_{R,T}(|\tilde{V}_t| + |\tilde{X}_t|) + C_{R,T}|\tilde{V}_t| \cdot \big[g\big(V_t(v)\big) + g\big(V_t(\hat{v})\big)\big],$$

where the non-negative function $g \in L^p(\mathbb{R}^d)$ with $p > d$.

Proof of strong well-posedness (cont)

Now, by Itô's formula, we get for every $t \leq T$,

$$
\begin{aligned}
&|\tilde{X}_{t\wedge\zeta_R}|^2 + |\tilde{V}_{t\wedge\zeta_R}|^2 \\
&\leq |x-\hat{x}|^2 + C_0|v-\hat{v}|^2 + C_{\lambda,R,T}\int_0^{t\wedge\zeta_R}\left(|\tilde{X}_s|^2 + |\tilde{V}_s|^2\right)ds \\
&\quad + C_{R,T}\int_0^{t\wedge\zeta_R}|\tilde{V}_s|^2 dA_s + \int_0^{t\wedge\zeta_R}\left[\sigma\left(V_s(\hat{v})\right) - \sigma\left(V_s(\hat{v})\right)\right]\tilde{V}_s dW_s,
\end{aligned}
$$

where $C_0 > 0$, and A_t is a continuous increasing process given by

$$
A_t := \int_0^t \Big(g\big(V_s(v)\big) + g\big(V_s(\hat{v})\big)\Big)^2 ds.
$$

Proof of strong well-posedness (cont)

In view of Lemma 2, we have that for every $\lambda > 0$,

$$\mathbb{E}e^{\lambda A_{t\wedge\zeta_R}} \leq C_{R,T}\|g\|_p e^{C_{R,T}} < \infty.$$

This in particular yields (7) by a result from Xie-Zhang (17).

Proof of ergodicity

By our assumption on F and the definition of $\mathcal{H}$, Lemma 4 in the case of **(A)** and Lemma 5 in the case of **(B)** to get that

$$\begin{aligned}
&\mathbb{E}\mathcal{H}(X_t, V_t) \\
&\leq \mathcal{H}(x, v) + c_1 t - c_2 \mathbb{E}\left(\int_0^t \mathcal{H}(X_s, V_s) ds\right) + c_3 \mathbb{E}\left(\int_0^t |G(V_s)|^2 ds\right) \\
&\leq \mathcal{H}(x, v) + c_4(1 + t) - c_2 \mathbb{E}\left(\int_0^t \mathcal{H}(X_s, V_s) ds\right).
\end{aligned}$$

By Gronwall's inequality, we have

$$\sup_{t \geq 0} \mathbb{E}\mathcal{H}(X_t, V_t) \leq C < \infty, \tag{8}$$

which implies the existence of invariant distributions.

Proof of ergodicity (cont)

One can show that the strong solution is strong Feller and irreducible. Thus, the exponential ergodicity follows by a standard argument.

Thank you!